# INFINITELY MANY NONRADIAL SINGULAR SOLUTIONS OF $\Delta u + e^u = 0$ IN $\mathbb{R}^N\backslash\{0\}$, $4 \le N \le 10$

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Abstract. We construct countably infinitely many nonradial singular solutions of the problem

$$\Delta u + e^u = 0 \text{ in } \mathbb{R}^N\backslash\{0\}, \quad 4 \le N \le 10$$

of the form

$$u(r,\sigma) = -2\log r + \log 2(N-2) + v(\sigma),$$

where $v(\sigma)$ depends only on $\sigma \in \mathbb{S}^{N-1}$. To this end we construct countably infinitely many solutions of

$$\Delta_{\mathbb{S}^{N-1}} v + 2(N-2)(e^v - 1) = 0, \quad 4 \le N \le 10,$$

using ODE techniques.

## 1. Introduction and Main results

We study singular solutions of the problem

$$\Delta U + e^U = 0 \text{ in } \mathbb{R}^N\backslash\{0\}. \tag{1.1}$$

This problem has the singular radial solution

$$U^*(R) = -2\log R + \log 2(N-2) \tag{1.2}$$

provided that $N \ge 3$. Let $\mathbb{R}_+ := \{x \in \mathbb{R};\ x > 0\}$, and let $\mathbb{S}^{N-1}$ denote the $(N-1)$-dimensional unit sphere. Let $(R,\sigma) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}$ be the spherical polar coordinates. We will find singular solutions of the form

$$U(R,\sigma) = -2\log R + \log 2(N-2) + V(\sigma). \tag{1.3}$$

The main result of the paper is the following:

**Theorem A.** *Suppose that $4 \le N \le 10$. The problem (1.1) has countably infinitely many nonradial singular solutions of the form (1.3). Here, $V(\sigma)$ is nonconstant.*

Substituting (1.3) into (1.1), we see that $V$ satisfies

$$\Delta_{\mathbb{S}^{N-1}} V + 2(N-2)(e^V - 1) = 0 \text{ on } \mathbb{S}^{N-1}, \tag{1.4}$$

where $\Delta_{\mathbb{S}^{N-1}}$ denotes the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. Theorem A immediately follows from Theorem B below.

**Theorem B.** *Suppose that $4 \le N \le 10$. The problem (1.4) has countably infinitely many axially symmetric nonconstant solutions.*

In this paper we mainly study (1.4).

When $N = 3$, Bidaut-Véron *et al.* [1] studied nonradial singular solutions of (1.1) and other equations. The equation (1.4) becomes $\Delta_{\mathbb{S}^2} V + 2(e^V - 1) = 0$. This is called the conformal Gaussian curvature equation, and this equation and related equations have

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been studied for more than three decades. All regular solutions of (1.4) are described in [2, 8]. In particular, axially symmetric solutions can be written explicitly as $V(\theta) = -2\log(\sqrt{c^2+1} - c\cos\theta)$, where $c \in \mathbb{R}$ is constant and $\theta \in [0,\pi]$ is the geodesic distance from the north pole of $\mathbb{S}^2$. Hence,

$$U(R,\theta) = -2\log R + \log 2 - \log(\sqrt{c^2+1} - c\cos\theta) \tag{1.5}$$

is a one parameter family of nonradial singular solutions of (1.1) in the case $N = 3$. The singular solution (1.5) can be seen as a singular solution of the Dirichlet problem

$$\begin{cases} \Delta U + e^U = 0 & \text{in } \Omega\backslash\{0\} \\ U = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega := \{U > 0\} \subset \mathbb{R}^3$.

Nontrivial one-point singular solutions of the equation $\Delta U + e^U = 0$ were constructed by several authors when the domain is bounded. In [11] Rébaï studied nonradial singular solutions in the case $N = 3$. Let $B_r$ denote the ball centered at the origin with radius $r > 0$. He showed, among other things, that there is small $\varepsilon > 0$ such that if $\xi_0 \in B_\varepsilon$, then the problem

$$\begin{cases} \Delta U + \lambda e^U = 0 & \text{in } B_1\backslash\{\xi_0\} \\ U = 0 & \text{on } \partial B_1 \end{cases}$$

has a singular solution for some $\lambda > 0$ provided that $N = 3$. In particular, this singular solution is not radially symmetric. Note that the same result was announced by H. Matano and his method is different from [11]. In [3] Dávila and Dupaigne constructed a singular solution when the domain is close to the unit ball provided that $N \geq 4$. Specifically, they showed that if $N \geq 4$ and $t > 0$ is small, then the problem

$$\begin{cases} \Delta U + \lambda e^U = 0 & \text{in } D_t\backslash\{\xi(t)\} \\ U = 0 & \text{on } \partial D_t \end{cases}$$

has a singular solution $(\lambda(t), U(x,t))$ such that as $t \to 0$,

$$\|U(x,t) - (-2\log|x - \xi(t)|)\|_{L^\infty(D_t)} \to 0 \qquad \text{and} \qquad \lambda(t) \to 2(N-2),$$

where $D_t := \{x + t\psi(x);\ x \in B_1 \subset \mathbb{R}^N\}$ and $\psi$ is a $C^2$-mapping from $\bar{B}_1$ to $\mathbb{R}^N$. Solutions with finitely or infinitely many singularities were constructed by Pacard [9] and Horshin [5, 6] when $N > 10$, and by Rébaï [10] when $N = 3$. Our solutions given by Theorem A are candidates of the asymptotic profiles of those singular solutions near a singular point.

Similar problems were studied for the equation $\Delta U + \lambda(1+U)^p = 0$ in [3, 11] and the equation $\Delta U + U^p = 0$ in [4]. In particular, Dancer *et al.* [4] constructed infinitely many nonradial positive singular solutions of the Lane-Emden equation

$$\Delta U + U^p = 0 \ \text{ in } \ \mathbb{R}^N\backslash\{0\} \tag{1.6}$$

with

$$\frac{N+1}{N-3} < p < p_{JL}(N-1) \quad \text{and} \quad N \geq 4, \tag{1.7}$$

where $p_{JL}(M)$ is defined by

$$p_{JL}(M) := \begin{cases} 1 + \frac{4}{M-4-2\sqrt{M-1}} & \text{if } M \geq 11, \\ \infty & \text{if } 2 \leq M \leq 10. \end{cases}$$

Let us consider the solution of the form

$$U(R,\sigma) = R^{-\frac{2}{p-1}} V(\sigma).$$

Then $V$ satisfies the equation

$$\Delta_{\mathbb{S}^{N-1}} V - \mu V + V^p = 0, \tag{1.8}$$

where

$$\mu := \frac{2}{p-1}\left(N-2-\frac{2}{p-1}\right).$$

If $p < (N+1)/(N-3)$, then all solutions are constant [1]. When $p = (N+1)/(N-3)$, $\mu = (N-3)(N-1)/4$. Then, (1.8) becomes Yamabe problem on $\mathbb{S}^{N-1}$ and various solutions are known. If (1.7) holds, then in [4] Dancer *et al.* showed that (1.8) has infinitely many nonconstant regular positive solutions. Theorem B in the present paper is its exponential counterpart.

Let us mention technical details. We construct nonconstant regular solutions of (1.4). We use an ODE approach. Specifically, we find solutions $v$ in the space of functions depending only on $\theta \in [0, \pi]$ which is the geodesic distance from the north pole of $\mathbb{S}^{N-1}$. Then $v$ satisfies

$$\begin{cases} v'' + (N-2)\frac{\cos\theta}{\sin\theta} v' + 2(N-2)(e^v - 1) = 0, & 0 < \theta < \pi, \\ v'(0) = v'(\pi) = 0. \end{cases} \tag{1.9}$$

If $v(\theta)$ satisfies (1.9), then $v(\pi - \theta)$ also satisfies (1.9). In order to make the problem easier we find symmetric solutions, i.e., $v(\theta) = v(\pi - \theta)$. Then (1.9) becomes the following:

$$\begin{cases} v'' + (N-2)\frac{\cos\theta}{\sin\theta} v' + 2(N-2)(e^v - 1) = 0, & 0 < \theta < \frac{\pi}{2}, \\ v'(0) = v'\left(\frac{\pi}{2}\right) = 0. \end{cases} \tag{1.10}$$

If $N \geq 4$, then (1.10) has the exact singular solution

$$v^*(\theta) := -2\log\sin\theta + \kappa_{N-1}, \quad \text{where} \quad \kappa_{N-1} := \log\frac{N-3}{N-2}. \tag{1.11}$$

In the study of the equation $\Delta U + e^U = 0$ defined in the Euclidean space it is well known that the transformation

$$X(T) = U(R) - U^*(R) \quad \text{and} \quad T := \log R$$

works well, where $U^*(R)$ is the singular solution defined by (1.2). Essentially the same transformation works well for the problem (1.10). Hereafter we consider the case $N \geq 4$. Then $v^*(\theta)$ is well defined. Let

$$x(t) := v(\theta) - v^*(\theta) \quad \text{and} \quad t := \log\tan\frac{\theta}{2}. \tag{1.12}$$

Then $x(t)$ satisfies

$$\begin{cases} x'' - (N-3)\tanh(t)x' + 2(N-3)(e^x - 1) = 0, & -\infty < t < 0, \\ \cosh(t)x'(t) + 2\sinh(t) \to 0 \quad \text{as} \quad t \to -\infty, \\ x'(0) = 0, \end{cases} \tag{1.13}$$

where we use the equality $v'(\theta) = \cosh(t)x'(t) + 2\sinh(t)$. The method so far is the same as the case $\Delta U + U^p = 0$ used in [4]. However, our method of the construction of solutions of (1.13) is different from that of [4] which uses the matched asymptotic expansions. Our proof is elementary and shorter. Using our method, one can obtain the main result of [4], i.e., the existence of infinitely many positive solutions of (1.8). See Section 4 of the present paper. We use a phase plane analysis in spite that (1.13) is not homogeneous. The effect of this inhomogeneity can be reduced by a scaling argument. A regular perturbation method and the winding number of the orbit $(x(t), x'(t))$ play important roles. This method is inspired by that of [7]. However, the authors of [7] used

a technical argument of the uniform convergence to the solution of the limit equation instead of a regular perturbation method.

This paper consists of four sections. In Section 2 we recall known results of radial solutions of $\Delta U + e^U = 0$. In Section 3 we prove Theorem B which leads to Theorem A. In Section 4 we briefly prove the existence of infinitely many positive radial solutions of (1.8), using our method.

## 2. Preliminaries

In this section we recall known results about the following equation of $\bar{x}(s)$:

$$\bar{x}'' + (N-3)\bar{x}' + 2(N-3)(e^{\bar{x}} - 1) = 0, \quad -\infty < s < \infty. \tag{2.1}$$

This is the limit equation of (1.13) as $t \to -\infty$ and it also appears in the problem

$$\begin{cases} \Delta U + e^U = 0 \quad \text{in } \mathbb{R}^{N-1}, \\ U \text{ is radial.} \end{cases} \tag{2.2}$$

First, we derive (2.1) from (2.2). We consider the initial value problem

$$\begin{cases} U'' + \frac{N-2}{R}U' + e^U = 0, \quad 0 < R < \infty, \\ U(0) = \bar{\alpha}, \\ U'(0) = 0. \end{cases} \tag{2.3}$$

The equation in (2.3) has the singular solution $U^*(R) = -2\log R + \bar{\kappa}_{N-1}$, where $\bar{\kappa}_{N-1} := \log 2(N-3)$. We define $\bar{x}(s)$ and $s$ by

$$\bar{x}(s) := U(R) - U^*(R) \qquad \text{and} \quad s := \log R,$$

respectively. The problem (2.3) becomes

$$\begin{cases} \bar{x}'' + (N-3)\bar{x}' + 2(N-3)(e^{\bar{x}} - 1) = 0, \qquad -\infty < s < \infty, \\ \bar{x}(s) - 2s + \bar{\kappa}_{N-1} - \bar{\alpha} \to 0 \ \text{ as } \ s \to -\infty, \\ e^{-s}(\bar{x}'(s) - 2) \to 0 \ \text{ as } \ s \to -\infty, \end{cases} \tag{2.4}$$

where we use the equalities $U(R) = \bar{x}(s) - 2s + \bar{\kappa}_{N-1}$ and $U'(R) = e^{-s}(\bar{x}'(s) - 2)$.

We use a phase plane argument. Let $\bar{y}(s) := \bar{x}'(s)$. By (2.4) we obtain the following:

$$\begin{cases} \bar{x}' = \bar{y}, & -\infty < s < \infty, \\ \bar{y}' = -(N-3)\bar{y} - 2(N-3)(e^{\bar{x}} - 1), & -\infty < s < \infty, \\ \bar{x}(s) - 2s + \bar{\kappa}_{N-1} - \bar{\alpha} \to 0 \ \text{ as } \ s \to -\infty, \\ e^{-s}(\bar{y}(s) - 2) \to 0 \ \text{ as } \ s \to -\infty. \end{cases} \tag{2.5}$$

Various properties of the solution $(\bar{x}(s), \bar{y}(s))$ of (2.5), which we call the orbit, are known. We summarize these properties of this orbit in the following proposition:

**Proposition 2.1.** *Assume that $N \geq 4$. The (2.5) has the unique entire solution. The orbit $\{(\bar{x}(s), \bar{y}(s));\ -\infty < s < \infty\}$ in the xy-plane starts along the line $y = 2$ at $s = -\infty$ and converges to the origin as $s \to \infty$. When $4 \leq N \leq 10$, the origin is a stable spiral and the orbit rotates clockwise around the origin. Therefore, there is $\{s_j\}_{j=1}^{\infty}$ ($s_1 < s_2 < \cdots \to \infty$) such that $\bar{y}(s_j) = 0$ ($s \in \{1, 2, \ldots\}$) and*

$$\bar{x}(s_2) < \bar{x}(s_4) < \cdots < \bar{x}(s_{2j}) < \cdots < 0 < \cdots < \bar{x}(s_{2j-1}) < \cdots < \bar{x}(s_3) < \bar{x}(s_1). \tag{2.6}$$

We briefly prove Proposition 2.1 for readers' convenience.

*Proof.* We omit the proof of the existence and uniqueness of the solution. We prove other properties of the orbit which are later used in the proof of the main theorem. Because of (2.5), $\lim_{s\to-\infty}(\bar{x}(s)-2s)=-\bar{\kappa}_{N-1}+\bar{\alpha}$ and $\lim_{s\to-\infty}(\bar{y}(x)-2)=0$. Thus the orbit starts along the line $y=2$ at $s\to-\infty$. The problem (2.5) has the Lyapunov function

$$E(x,y):=\frac{y^2}{2}+2(N-3)(e^x-x). \tag{2.7}$$

Indeed, we have

$$\frac{d}{ds}E(\bar{x}(s),\bar{y}(s))=-(N-3)\bar{y}^2(s)\le 0. \tag{2.8}$$

We show by contradiction that the problem

$$\begin{cases}\bar{x}'=\bar{y}\\ \bar{y}'=-(N-3)\bar{y}-2(N-3)(e^{\bar{x}}-1)\end{cases} \tag{2.9}$$

does not have a nontrivial periodic orbit. Assume that (2.9) has a nontrivial periodic orbit. Then we see by (2.8) that $\bar{y}(s)\equiv 0$. Because of (2.9), $\bar{x}'(s)\equiv 0$ and $\bar{x}(s)$ is constant. The orbit $(\bar{x}(s),\bar{y}(s))$ is an equilibrium of (2.9). This contradicts the assumption. Hence, (2.9) does not have a nontrivial periodic orbit.

Let

$$\Omega_c:=\{(x,y);\ E(x,y)<c\}. \tag{2.10}$$

For each large $c>0$, $\Omega_c$ is a bounded set in the $xy$-plane. For large $c>0$, there is $s_0\in\mathbb{R}$ such that $\{(\bar{x}(s),\bar{y}(s))\}_{s\ge s_0}\subset\Omega_c$, and $\{(\bar{x}(s),\bar{y}(s))\}_{s\ge s_0}$ is bounded. Because (2.9) does not have a periodic orbit, by the Poincaré-Bendixson theorem we see that the orbit converges to the origin which is the unique equilibrium.

In order to study the behavior of the orbit near the origin we consider the linearized problem of (2.9) at the origin, i.e.,

$$\begin{pmatrix}0 & 1\\ -2(N-3) & -(N-3)\end{pmatrix}.$$

The two eigenvalues $\lambda_\pm$ of the matrix are given by the characteristic equation $\lambda^2+(N-3)\lambda+2(N-3)=0$. We have

$$\lambda_\pm:=\frac{1}{2}\left\{-(N-3)\pm\sqrt{(N-3)(N-11)}\right\}.$$

If $4\le N\le 10$, then the two eigenvalues are complex with negative real part. Hence, the origin is a stable spiral. We see by the direction of the vector field defined by (2.9) that the orbit rotates clockwise around the origin and (2.6) holds. The proof is complete. □

## 3. Proof of Theorem B

In order to find solutions of (1.10) we study the problem

$$\begin{cases}v''+(N-2)\frac{\cos\theta}{\sin\theta}v'+2(N-2)(e^v-1)=0, & 0<\theta<\frac{\pi}{2},\\ v(0)=\alpha,\\ v'(0)=0,\end{cases} \tag{3.1}$$

where $\alpha\in\mathbb{R}$ is a parameter. We call $v(\theta)$ the solution of (3.1) if

$$v(\theta)\in C^2(0,\frac{\pi}{2}]\cap C^1[0,\frac{\pi}{2}] \tag{3.2}$$

and if $v(\theta)$ satisfies (3.1). We also study the problem

$$\begin{cases} v''+(N-2)\frac{\cos\theta}{\sin\theta}v'+2(N-2)(e^v-1)=0, & 0<\theta<\frac{\pi}{2}, \\ \lim_{\theta\downarrow 0}v(\theta)=\alpha, \\ \lim_{\theta\downarrow 0}v'(\theta)=0. \end{cases} \tag{3.3}$$

We call $v(\theta)$ a solution of (3.3) if

$$v(\theta)\in C^2(0,\frac{\pi}{2}] \tag{3.4}$$

and if $v(\theta)$ satisfies (3.3). If $v(\theta)$ is the solution of (3.1), then the restricted function of $v(\theta)$ is the solution of (3.3). We assume that $v(\theta)$ is the solution of (3.3). We define $v(0)=\alpha$ so that $v(\theta)$ is defined on $[0,\frac{\pi}{2}]$ and continuous at $\theta=0$. Using L'Hospital's rule, we have

$$v'(0)=\lim_{\theta\downarrow 0}\frac{v(\theta)-\alpha}{\theta}=\lim_{\theta\downarrow 0}\frac{v'(\theta)}{1}=0.$$

Since $v'(0)=0=\lim_{\theta\downarrow 0}v'(\theta)$, $v'(\theta)\in C^0[0,\frac{\pi}{2}]$. We see that $v(\theta)\in C^1[0,\frac{\pi}{2}]$ and $v(\theta)$ is the solution of (3.1). Thus, the solution of (3.3) can be uniquely extended as the solution of (3.1). The problem (3.1) is equivalent to (3.3). Hence, we consider (3.3).

**Lemma 3.1.** *Let $x(t)$ be defined by (1.12). The function $v(\theta)$ is the solution of (3.3) if and only if*

$$x(t)\in C^2(-\infty,0] \tag{3.5}$$

*and $x(t)$ satisfies*

$$\begin{cases} x''-(N-3)\tanh(t)x'+2(N-3)(e^x-1)=0, & -\infty<t<0, \\ x(t)+2\log\cosh(t)+\kappa_{N-1}-\alpha\to 0 \quad \textit{as} \quad t\to-\infty, \\ \cosh(t)x'(t)+2\sinh(t)\to 0 \quad \textit{as} \quad t\to-\infty. \end{cases} \tag{3.6}$$

*Proof.* By direct calculation we see that the equation in (3.3) is equivalent to that of (3.6). It follows from the definition of $x(t)$ that (3.5) holds if and only if (3.4). Using (1.12) and the equality $v'(\theta)=\cosh(t)x'(t)+2\sinh(t)$, we see that the initial conditions of (3.3) are equivalent to those of (3.6). The proof is complete. $\square$

We call $x(t)$ the solution of (3.6) if (3.5) and (3.6) hold.

If the solution of (3.6) satisfies $x'(0)=0$, then this solution satisfies (1.13). Therefore, the function $v(\theta)$, which is associated to $x(t)$ by (1.12), becomes a solution of (1.10).

**Lemma 3.2.** *Let*

$$\tilde{x}(s):=x(t) \quad \textit{and} \quad s:=t+\frac{\alpha}{2}. \tag{3.7}$$

*The function $x(t)$ is a solution of (3.6) if and only if*

$$\tilde{x}(s)\in C^2(-\infty,\frac{\alpha}{2}] \tag{3.8}$$

*and $\tilde{x}(s)$ satisfies*

$$\begin{cases} \tilde{x}''-(N-3)\tanh\left(s-\frac{\alpha}{2}\right)\tilde{x}'+2(N-3)(e^{\tilde{x}}-1)=0, & -\infty<s<\frac{\alpha}{2}, \\ \tilde{x}(s)-2s-2\log 2+\kappa_{N-1}\to 0 \quad \textit{as} \quad s\to-\infty, \\ e^{-s}(\tilde{x}'(s)-2)\to 0 \quad \textit{as} \quad s\to-\infty. \end{cases} \tag{3.9}$$

*Proof.* It is clear that the equation in (3.6) is equivalent to that in (3.9). We check the equivalence of initial conditions. Let $x(t)$ be a solution of (3.6). Since

$$\lim_{s\to-\infty}\left(2\log\cosh\left(s-\frac{\alpha}{2}\right)-2\log\frac{e^{-s}}{2}-\alpha\right)=\lim_{s\to-\infty}2\log(e^{2s-\alpha}+1)=0,$$

$$\begin{aligned}0&=\lim_{t\to-\infty}(x(t)+2\log\cosh(t)+\kappa_{N-1}-\alpha)\\&=\lim_{s\to-\infty}\left\{\tilde{x}(s)+2\log\frac{e^{-s}}{2}+\kappa_{N-1}+\left(2\log\cosh\left(s-\frac{\alpha}{2}\right)-2\log\frac{e^{-s}}{2}-\alpha\right)\right\}\\&=\lim_{s\to-\infty}\left(\tilde{x}(s)-2s-2\log 2+\kappa_{N-1}\right).\end{aligned}$$

Since

$$0=\lim_{t\to-\infty}(\cosh(t)\tilde{x}'(t)+2\sinh(t))=\lim_{s\to-\infty}\left\{\frac{1}{2}(e^{-s+\frac{\alpha}{2}}+e^{s-\frac{\alpha}{2}})(\tilde{x}'(s)-2)+2e^{s-\frac{\alpha}{2}}\right\},$$

$$0=\lim_{s\to-\infty}\left|\frac{1}{2}(e^{-s+\frac{\alpha}{2}}+e^{s-\frac{\alpha}{2}})(\tilde{x}'(s)-2)\right|\geq\frac{e^{\frac{\alpha}{2}}}{2}\lim_{s\to-\infty}|e^{-s}(\tilde{x}'(s)-2)|.$$

Hence, $\lim_{s\to-\infty}e^{-s}(\tilde{x}'(s)-2)=0$. Thus $\tilde{x}(s)$ satisfies (3.9). We can check that the converse is also true. We omit the detail. The proof is complete. □

We call $\tilde{x}(s)$ the solution of (3.9) if (3.8) and (3.9) hold.

Let $\tilde{x}(s)$ be the solution of (3.9), and let $\tilde{y}(s):=\tilde{x}'(s)$. The pair of functions $(\tilde{x}(s),\tilde{y}(s))$ satisfies

$$\begin{cases}\tilde{x}'=\tilde{y}, & -\infty<s<s_0,\\ \tilde{y}'=(N-3)\tanh\left(s-\frac{\alpha}{2}\right)\tilde{y}-2(N-3)(e^{\tilde{x}}-1), & -\infty<s<s_0,\\ \tilde{x}(s)-2s+\tilde{\kappa}_{N-1}\to 0\quad\text{as}\quad s\to-\infty,\\ e^{-s}(\tilde{y}(s)-2)\to 0\quad\text{as}\quad s\to-\infty,\end{cases}\tag{3.10}$$

where $\tilde{\kappa}_{N-1}:=\kappa_{N-1}-2\log 2$. We study the behavior of the orbit $(\tilde{x}(s),\tilde{y}(s))$ when $\alpha$ is large. Since $\alpha$ is large, $\tanh\left(s-\frac{\alpha}{2}\right)$ is close to $-1$. We can expect that $(\tilde{x}(s),\tilde{y}(s))$ behaves like the solution of (2.5) with $\bar{\alpha}=\bar{\kappa}_{N-1}-\kappa_{N-1}+2\log 2$.

**Lemma 3.3.** *Let $s_0\in\mathbb{R}$ be fixed. Let*

$$u(r):=\tilde{x}(s)-2s+\tilde{\kappa}_{N-1}\quad\textit{and}\quad r:=e^s.$$

*The pair of functions*

$$(\tilde{x}(s),\tilde{y}(s))\in C^2(-\infty,s_0]\times C^1(-\infty,s_0]\tag{3.11}$$

*satisfies (3.10) if*

$$u(r)\in C^2(0,r_0]\cap C^1[0,r_0]\tag{3.12}$$

*and $u(r)$ satisfies the problem*

$$\begin{cases}u''+\frac{N-2}{r}u'+8(N-2)e^u-\frac{2(N-3)\delta}{1+\delta r^2}(ru'+2)=0, & 0<r\leq r_0,\\ u(0)=0,\\ u'(0)=0,\end{cases}\tag{3.13}$$

*where $r_0:=e^{s_0}$ and $\delta=e^{-\alpha}$.*

*Proof.* The proof is almost the same as that of Lemma 3.1. We omit the proof. □

Lemma 3.3 shows that the limit equation of (3.13) as $\delta\downarrow 0$ is $u''+\frac{N-2}{r}u'+8(N-2)e^u=0$. We call $u(r)$ the solution of (3.13) if (3.12) and (3.13) hold.

**Lemma 3.4.** *Let $r_0>0$ be fixed. Let $u(r,\delta)$ be the solution of (3.13). For each small $\varepsilon>0$, there exists $\delta_0>0$ such that if $|\delta|<\delta_0$, then $\|u(\,\cdot\,,0)-u(\,\cdot\,,\delta)\|_{C^1[0,r_0]}<\varepsilon$.*

*Proof.* We define $\mathcal{F}(u,\delta)$ and $\mathcal{G}(u,\delta)$ by

$$\begin{aligned}\mathcal{F}(u(r),\delta)&:=\int_0^r\left(\frac{1}{s^{N-2}}\int_0^s\mathcal{G}(u(\tau),\delta)d\tau+\frac{2(N-3)\delta s}{1+\delta s^2}u(s)\right)ds\quad\text{and}\\ \mathcal{G}(u(r),\delta)&:=-8(N-2)r^{N-2}e^{u(r)}-2(N-3)\delta\frac{N-1+(N-3)\delta r^2}{(1+\delta r^2)^2}r^{N-2}u(r)\\ &\qquad+\frac{4(N-3)\delta}{1+\delta r^2}r^{N-2},\end{aligned}$$

respectively. First, we show that if $u\in C^0[0,r_0]$ satisfies

$$u(r)=\mathcal{F}(u(r),\delta)\quad\text{for}\quad r\in[0,r_0],\tag{3.14}$$

then $u$ is a solution of (3.13). Let $u\in C^0[0,r_0]$ be a function such that (3.14) holds. We immediately see that $u(0)=0$. Since $\mathcal{F}(u(r),\delta)\in C^1(0,r_0]$, $u\in C^1(0,r_0]$. Differentiating (3.14) with respect to $r$, we have

$$u'(r)=\frac{1}{r^{N-2}}\int_0^r\mathcal{G}(u(\tau),\delta)d\tau+\frac{2(N-3)\delta r}{1+\delta r^2}u(r).\tag{3.15}$$

Since the right-hand side of (3.15) is in $C^1(0,r_0]$, $u\in C^2(0,r_0]$. Multiplying (3.15) by $r^{N-2}$ and differentiating it with respect to $r$, we have

$$\begin{aligned}(r^{N-2}u')'&=\mathcal{G}(u(r),\delta)+\left(\frac{2(N-3)\delta r^{N-1}}{1+\delta r^2}u(r)\right)'\\ &=-8(N-2)r^{N-2}e^u+\frac{2(N-3)\delta}{1+\delta r^2}r^{N-2}(ru'(r)+2).\end{aligned}$$

Thus, $u$ satisfies the equation in (3.13). There is $C>0$ such that $|\mathcal{G}(u(\tau),\delta)|\le C\tau^{N-2}$ and $\left|\frac{2(N-3)\delta r}{1+\delta r^2}u(r)\right|\le Cr$, we have

$$\left|\frac{1}{r^{N-2}}\int_0^r\mathcal{G}(u(\tau),\delta)d\tau+\frac{2(N-3)\delta r}{1+\delta r^2}u(r)\right|\le C\left(\frac{1}{N-1}+1\right)r\to0\quad(r\downarrow0).$$

Therefore, by (3.15) we see that $\lim_{r\downarrow0}u'(r)=0$. On the other hand, by L'Hospital's rule we have

$$u'(0)=\lim_{r\downarrow0}\frac{u(r)-0}{r}=\lim_{r\downarrow0}\frac{u'(r)}{1}=0.$$

Since $u'(0)=0=\lim_{r\downarrow0}u'(r)$, $u\in C^1[0,r_0]$. Since $u\in C^2(0,r_0]\cap C^1[0,r_0]$, $u$ is the solution of (3.13).

Let $\mathcal{H}(u,\delta):=u-\mathcal{F}(u,\delta)$. We study the solution of the problem

$$\mathcal{H}(u,0)=0\quad\text{in}\quad C^0[0,r_0].\tag{3.16}$$

Then (3.16) is equivalent to (3.13) with $\delta=0$. Let $U(R):=u(r)+\log8(N-2)$ and $R:=r$. The problem (3.13) with $\delta=0$ is equivalent to the problem (2.3) with $\bar\alpha=\log8(N-2)$. It is well known that (2.3) has the unique solution $U(R)$ which defined in $R\ge0$. Hence, (3.16) has the unique solution $u_0(r)\in C^0[0,r_0]$. We have

$$\mathcal{H}(u_0,0)=0\quad\text{in}\quad C^0[0,r_0].\tag{3.17}$$

We consider the linearized problem

$$(D_u\mathcal{H}(u_0(r),0)[\phi(r)]=)\phi(r)-D_u\mathcal{F}(u_0(r),0)[\phi(r)]=0,$$

where $\phi \in C^0[0, r_0]$. Since

$$D_u\mathcal{F}(u_0, 0)[\phi] = \int_0^r \frac{1}{s^{N-1}} \int_0^s (-8(N-2)\tau^{N-2} e^{u_0(\tau)}\phi(\tau)) d\tau ds,$$

we can by a similar argument show that $\phi \in C^2(0, r_0] \cap C^1[0, r_0]$ and $\phi$ is the solution of the problem

$$\begin{cases} \phi'' + \frac{N-2}{r}\phi' + 8(N-2)e^{u_0}\phi = 0, & 0 < r \le r_0, \\ \phi(0) = 0, \\ \phi'(0) = 0. \end{cases} \tag{3.18}$$

Because of the uniqueness of the solution of (3.18), $\phi(r) \equiv 0$ $(0 \le r \le r_0)$. Since $D_u\mathcal{F} : C^0[0, r_0] \to C^0[0, r_0]$ is compact, by the Fredholm alternative we see that

$$\text{the mapping } D_u\mathcal{H}(u_0, 0)[\phi] = \phi - D_u\mathcal{F}(u_0, 0)[\phi] \text{ is invertible.} \tag{3.19}$$

We find a solution near $u_0$. It is clear that

$$\mathcal{H}(u, \delta) \text{ is } C^1 \text{ near } (u_0, 0). \tag{3.20}$$

By (3.17), (3.19), and (3.20) we apply the implicit function theorem to $\mathcal{H}(u, \delta) = 0$ at $(u_0, 0)$. There is a small $\delta_0 > 0$ and a smooth mapping $u = u_\delta$ such that if $|\delta| < \delta_0$, then $\mathcal{H}(u_\delta, \delta) = 0$ in $C^0[0, r_0]$ and $\|u_0 - u_\delta\|_{C^0[0,r_0]} \to 0$ as $\delta \to 0$. For each small $\eta_0 > 0$, there is $\delta > 0$ such that $|\mathcal{G}(u_0(r), 0) - \mathcal{G}(u_\delta(r), \delta)| \le \eta_0 r^{N-2}$ for $r \in [0, r_0]$. Using this inequality and (3.15), we have

$$\begin{aligned} |u_0'(r) - u_\delta'(r)| &\le \frac{1}{r^{N-2}} \int_0^{r_0} \eta_0 r^{N-2} dr + \delta \left\| \frac{2(N-3)r}{1+\delta r^2} u_\delta(r) \right\|_{C^0[0,r_0]} \\ &= \frac{\eta_0 r_0}{N-1} + \delta \left\| \frac{2(N-3)r}{1+\delta r^2} u_\delta(r) \right\|_{C^0[0,r_0]}. \end{aligned} \tag{3.21}$$

The inequality (3.21) means that $\|u_0' - u_\delta'\|_{C^0[0,r_0]} \to 0$ as $\delta \downarrow 0$. Thus, $\|u_0 - u_\delta\|_{C^1[0,r_0]} \to 0$ as $\delta \downarrow 0$ and the proof is complete. $\square$

**Lemma 3.5.** *Let $(\bar{x}(s), \bar{y}(s))$ be the solution of (2.5) with $\bar{\alpha} = \log 8(N-2)$, and let $(\tilde{x}(s), \tilde{y}(s))$ be the solution of (3.10). For each $s_0 > 0$ and $\varepsilon > 0$, there is $\alpha_0 > 0$ such that if $\alpha > \alpha_0$, then*

$$\|\bar{x}(\cdot) - \tilde{x}(\cdot)\|_{C^0(-\infty,s_0]} < \varepsilon \qquad \text{and} \qquad \|\bar{y}(\cdot) - \tilde{y}(\cdot)\|_{C^0(-\infty,s_0]} < \varepsilon. \tag{3.22}$$

*Proof.* Let $U(R) := \bar{x}(s) - 2s + \bar{\kappa}_{N-1}$ and $R := e^s$. Then $U(R)$ satisfies (2.3) with $\bar{\alpha} = \log 8(N-2)$. Let $u_0(r) := U(R) - \log 8(N-2)$ and $r := R$. Then $u_0(r)$ satisfies (3.13) with $\delta = 0$.

Let $u_\delta(r) := \tilde{x}(s) - 2s + \tilde{\kappa}_{N-1}$ and $r := e^s$. Then $u_\delta(r)$ satisfies (3.13). Because of Lemma 3.4, for small $\varepsilon > 0$, there is $\delta_0 > 0$ such that

$$\text{if } |\delta| < \delta_0, \text{ then } \|u_0(\cdot) - u_\delta(\cdot)\|_{C^0[0,r_0]} < \varepsilon \text{ and } \|u_0'(\cdot) - u_\delta'(\cdot)\|_{C^0[0,r_0]} < e^{-s_0}\varepsilon, \tag{3.23}$$

where $s_0 := \log r_0$. Since $u_0(r) = \bar{x}(s) - 2s + \bar{\kappa}_{N-1} - \log 8(N-2)$ and $u_\delta(r) = \tilde{x}(s) - 2s + \tilde{\kappa}_{N-1}$, we have

$$\|u_0(\cdot) - u_\delta(\cdot)\|_{C^0(0,r_0]} = \|\bar{x}(\cdot) - \tilde{x}(\cdot)\|_{C^0(-\infty,s_0]},$$

where we use the equality $\bar{\kappa}_{N-1} - \log 8(N-2) - \tilde{\kappa}_{N-1} = 0$. Since $u_0'(r) = e^{-s}(\bar{x}'(s) - 2)$ and $u_\delta'(r) = e^{-s}(\tilde{x}'(s) - 2)$,

$$\|u_0'(\cdot) - u_\delta'(\cdot)\|_{C^0(0,r_0]} = \left\| e^{-s}(\bar{x}'(\cdot) - \tilde{x}'(\cdot)) \right\|_{C^0(-\infty,s_0]} \ge e^{-s_0} \left\| \bar{x}'(\cdot) - \tilde{x}'(\cdot) \right\|_{C^0(-\infty,s_0]} \tag{3.24}$$

Let $\alpha_0 := -\log \delta_0$. Combining (3.24) and (3.23), we see that (3.22) holds for $\alpha > \alpha_0$. $\square$

Hereafter, by $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ we denote the solution of (3.10). By $(\bar{x}(s),\bar{y}(s))$ we denote the solution of (2.5) with $\bar{\alpha}:=\log 8(N-2)$. Lemma 3.5 says that for each fixed $s_0\in\mathbb{R}$, the orbit $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ is close to $(\bar{x}(s),\bar{y}(s))$ for $s\in(-\infty,s_0]$ if $\alpha$ is large. Since $(\bar{x}(s),\bar{y}(s))$ rotates around the origin infinitely many times (Proposition 2.1), we expect that there are infinitely many $\alpha\in\mathbb{R}$ such that $\tilde{y}(\frac{\alpha}{2},\alpha)=0$. Since $y(0,\alpha)=\tilde{y}(\frac{\alpha}{2},\alpha)=0$, $u(r)$ corresponding to $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ is the solution of (1.10). To prove the existence of such $\alpha$ we use the "half winding numbers" of the two orbits $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ and $(\bar{x}(s),\bar{y}(s))$. We define $\tilde{W}_I(\alpha)$ and $\bar{W}_I$ by

$$\tilde{W}_I(\alpha):=\sharp\{s\in I;\ \tilde{y}(s,\alpha)=0\},$$
$$\bar{W}_I:=\sharp\{s\in I;\ \bar{y}(s)=0\},$$

where $I\subset\mathbb{R}$ is an interval. For example, it is clear that $\bar{W}_{(-\infty,s_1]}<\infty$ if $|s_1|<\infty$. Proposition 2.1 says that $\bar{W}_{(-\infty,\infty)}=\infty$.

**Lemma 3.6.** *Let $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ be the solution of (3.10). There is a sequence $\{\alpha_j\}_{j=1}^{\infty}$ $(\alpha_1<\alpha_2<\cdots\to+\infty)$ such that*

$$\tilde{W}_{(-\infty,\frac{\alpha_1}{2}]}(\alpha_1)<\tilde{W}_{(-\infty,\frac{\alpha_2}{2}]}(\alpha_2)<\cdots\to\infty \tag{3.25}$$

*and*

$$\tilde{y}(\frac{\alpha_j}{2},\alpha_j)=0\quad\text{for}\quad j\in\{1,2,\ldots\}.$$

*Proof.* First, we show that if $\alpha$ is large, then

$$\tilde{W}_{(-\infty,\frac{\alpha}{2}]}(\alpha)<\infty. \tag{3.26}$$

Since $(0,0)$ is the equilibrium of the vector filed defined by the first order system in (3.10), we see that $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))\neq(0,0)$ for $s\in\mathbb{R}$. If $(\tilde{x},\tilde{y})$ is on the $x$-axis, then by (3.10) we have

$$\begin{cases}\tilde{x}'=0\\ \tilde{y}'=-2(N-3)(e^{\tilde{x}}-1).\end{cases}$$

Therefore, $\tilde{y}'\neq 0$, since $\tilde{x}\neq 0$. This means that the orbit does not stay on the $x$-axis when it crosses the $x$-axis and that $\tilde{W}_{(-\infty,s]}(\alpha)$ increases by one whenever the orbit crosses the $x$-axis. Let $s_0\in\mathbb{R}$ be fixed. We take a large $\alpha>0$ such that $s_0<\frac{\alpha}{2}$. The orbit $(\bar{x}(s),\bar{y}(s))$ $(-\infty<s<s_0)$ rotates around the origin finitely many times. Because of Lemma 3.5, $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ is close to $(\bar{x}(s),\bar{y}(s))$ for $s\in(-\infty,s_0)$, and hence $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ $(-\infty<s<s_0)$ rotates around the origin finitely many times. We see that $\tilde{W}_{(-\infty,s_0)}(\alpha)<\infty$. We study the behavior of the orbit $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ in the interval $[s_0,\frac{\alpha}{2}]$. Let $E(x,y)$ be as defined by (2.7). Then, for $s\le\frac{\alpha}{2}$,

$$\begin{aligned}\frac{d}{ds}E(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))&=\tilde{y}(\tilde{y}'+2(N-3)(e^{\tilde{x}}-1))\\&=(N-3)\tanh\left(s-\frac{\alpha}{2}\right)\tilde{y}^2\\&\le 0.\end{aligned}$$

Thus, $(\tilde{x}(s,\alpha),\tilde{y}(s,\alpha))$ $(s_0\le s\le\frac{\alpha}{2})$ is in the bounded set $\Omega_{c_0}\subset\mathbb{R}^2$, where $\Omega_c$ is defined by (2.10) and $c_0:=E(\tilde{x}(s_0,\alpha),\tilde{y}(s_0,\alpha))$. In particular, $\tilde{x}(s,\alpha)$ $(s_0\le s\le\frac{\alpha}{2})$ is bounded. On the other hand, $\tilde{x}(s,\alpha)$ satisfies the following linear ODE of the second order:

$$\tilde{x}''-(N-3)\tanh\left(s-\frac{\alpha}{2}\right)\tilde{x}'+2(N-3)V(\tilde{x})\tilde{x}=0,$$

where

$$V(\tilde{x}):=\begin{cases}\frac{e^{\tilde{x}}-1}{\tilde{x}} & \text{if } \tilde{x}\neq 0,\\ 1 & \text{if } \tilde{x}=0.\end{cases}$$

Since the interval $[s_0, \frac{\alpha}{2}]$ is compact, $\tilde{x}(s,\alpha)$ has at most finitely many critical points in $[s_0, \frac{\alpha}{2}]$, and hence $\tilde{y}(s,\alpha)$ also has at most finitely many zeros in $[s_0, \frac{\alpha}{2}]$. Thus, $\tilde{W}_{[s_0,\frac{\alpha}{2}]} < \infty$. Since $\tilde{W}_{(-\infty,\frac{\alpha}{2}]}(\alpha) = \tilde{W}_{(-\infty,s_0)}(\alpha) + \tilde{W}_{[s_0,\frac{\alpha}{2}]}(\alpha)$, $\tilde{W}_{(-\infty,\frac{\alpha}{2}]}(\alpha) < \infty$.

Second, we show that

$$\tilde{W}_{(-\infty,\frac{\alpha}{2}]}(\alpha) \to \infty \quad \text{as} \quad \alpha \to \infty. \tag{3.27}$$

Since $\bar{W}_{(-\infty,\infty)} = \infty$, we see the following: For each large $M > 0$, there are a large $s_0 \in \mathbb{R}$ and a large $\alpha_0 (> 2s_0)$ such that if $\alpha > \alpha_0$, then $\tilde{W}_{(-\infty,s_0)}(\alpha) > M$. Since $\tilde{W}_{(-\infty,\frac{\alpha}{2}]}(\alpha) \geq \tilde{W}_{(-\infty,s_0)}(\alpha)$, (3.27) holds.

It is clear that $(\tilde{x}(\frac{\alpha}{2},\alpha), \tilde{y}(\frac{\alpha}{2},\alpha))$ is continuous in $\alpha$. Because of (3.27), (3.26), and this continuity, there is a sequence $\{\alpha_j\}_{j=1}^{\infty}$ ($\alpha_1 < \alpha_2 < \cdots \to \infty$) such that $\tilde{y}(\frac{\alpha_j}{2}, \alpha_j) = 0$. We can choose a subsequence, which is still denoted by $\{\alpha_j\}_{j=1}^{\infty}$, such that (3.25) holds, because of (3.27) and (3.26). The proof is complete. □

We are in a position to prove Theorem B.

*Proof of Theorem B.* Let $\{\alpha_j\}_{j=1}^{\infty}$ be a sequence given in Lemma 3.6, and let $(\tilde{x}(s,\alpha), \tilde{y}(s,\alpha))$ be the solution of (3.10). We let

$$x(t,\alpha_j) := \tilde{x}(s,\alpha_j), \quad y(t,\alpha_j) := \tilde{y}(s,\alpha_j), \quad \text{and} \quad t := s - \frac{\alpha_j}{2}.$$

Then $(x(t,\alpha_j), y(t,\alpha_j))$ is a solution of (3.6) and $x'(0,\alpha_j) = y(0,\alpha_j) = 0$. Let $v(\theta,\alpha_j)$ be defined by (1.12) with $x(t) = x(t,\alpha_j)$. Since $v'(\theta,\alpha_j) = \cosh(t)x'(t,\alpha_j) - 2\sinh(t)/\cosh(t)$, $v'(\frac{\pi}{2},\alpha_j) = 0$, and hence $v(\theta,\alpha_j)$ satisfies (1.10). We define $\tilde{v}(\theta,\alpha_j)$ by

$$\tilde{v}(\theta,\alpha_j) := \begin{cases} v(\theta,\alpha_j) & 0 \leq \theta \leq \frac{\pi}{2}, \\ v(\pi-\theta,\alpha_j) & \frac{\pi}{2} < \theta \leq \pi. \end{cases}$$

Then $\tilde{v}(\theta,\alpha_j)$ ($j \in \{1,2,\ldots\}$) is a classical solution of (1.9). The proof is complete. □

## 4. Infinitely many radial solutions of (1.8)

In this section we briefly prove the following:

**Proposition 4.1** ([4, Theorem 1.1])**.** *If (1.7) holds, then (1.8) has infinitely many positive radial solutions. Therefore, (1.6) has infinitely many singular positive nonradial solutions.*

*Proof.* In order to find radial solutions of (1.8) we study

$$\begin{cases} v'' + (N-2)\frac{\cos\theta}{\sin\theta}v - \mu v + v^p = 0, & 0 < \theta < \frac{\pi}{2}, \\ v(0) = \alpha, \\ v'(0) = 0. \end{cases} \tag{4.1}$$

We find $\alpha > 0$ such that $v'(\frac{\pi}{2}) = 0$. We define

$$q := \frac{2}{p-1}, \quad A := \{q(N-3-q)\}^{\frac{1}{p-1}}, \quad \text{and} \quad m := A^{-\frac{p-1}{2}}.$$

By direct calculation we see that $v^*(\theta) := A\sin^{-q}\theta$ is a singular solution of the equation in (4.1). Using the transformation

$$x(t) := \frac{v(\theta)}{v^*(\theta)} \quad \text{and} \quad t := \frac{1}{m}\log\tan\frac{\theta}{2},$$

we have

$$\begin{cases} x''-(N-3-2q)m\tanh(mt)x'-x+x^p=0, & -\infty<t<0,\\ x(t)A\cosh^q(mt)\to\alpha \ \text{ as } \ t\to-\infty, \\ \cosh(mt)\frac{d}{dt}\left\{x(t)A\cosh^q(mt)\right\}\to 0 \ \text{ as } \ t\to-\infty. \end{cases}$$

We use the change of variables

$$\tilde{x}(s):=x(t) \quad \text{and} \quad s:=t+\frac{1}{mq}\log\alpha$$

which corresponding to (3.7). We have

$$\begin{cases} \tilde{x}'=\tilde{y},\\ \tilde{y}'=(N-3-2q)m\tanh(ms-\frac{\log\alpha}{q})\tilde{x}'+\tilde{x}-\tilde{x}^p,\\ \tilde{x}(s)A\cosh^q(ms)\to 1 \ \text{ as } \ s\to-\infty,\\ \cosh(ms)\frac{d}{ds}\left\{\tilde{x}(s)A\cosh^q(ms)\right\}\to 0 \ \text{ as } \ s\to-\infty. \end{cases} \tag{4.2}$$

We let $\delta:=\alpha^{-\frac{4}{q}}$. We consider the case where $\alpha$ is large. Then $\delta>0$ is small. Let

$$\frac{u(r)}{u^*(r)}=\tilde{x}(s), \quad r=e^{ms}, \quad \text{and} \quad u^*(r):=Ar^{-q}.$$

Then by (4.2) we have

$$\begin{cases} u''+\frac{N-2}{r}u'+u^p-\frac{2(N-3-2q)\delta}{1+\delta r^2}(ru'+qu)=0,\\ u(0)=1,\\ u'(0)=0. \end{cases}$$

Note that the limit equation as $\delta\to 0$ is $u''+\frac{N-2}{r}u'+u^p=0$. Using the same argument as in Lemmas 3.4 and 3.5, we can show that for each $s_0\in\mathbb{R}$ if $\alpha$ is large, then the orbit $(\tilde{x}(s),\tilde{y}(s))$ of (4.2) is close to the orbit $(\bar{x}(s),\bar{y}(s))$ of

$$\begin{cases} \bar{x}'=\bar{y},\\ \bar{y}'=-(N-3-2q)m\bar{y}+\bar{x}-\bar{x}^p,\\ \bar{x}(s)A\cosh^q(ms)\to 1 \ \text{ as } \ s\to-\infty,\\ \cosh(ms)\frac{d}{ds}\left\{\bar{x}(s)A\cosh^q(ms)\right\}\to 0 \ \text{ as } \ s\to-\infty. \end{cases} \tag{4.3}$$

in the interval $(-\infty,s_0]$. It is well known that the orbit $(\bar{x}(s),\bar{y}(s))$ is a heteroclinic orbit between the two equilibria $(0,0)$ and $(1,0)$ under the condition where $N-3-2q>0$. The linearization of (4.3) at $(1,0)$ is

$$\begin{pmatrix} 0 & 1\\ 1-p & -(N-3-2q)m \end{pmatrix}.$$

The two eigenvalues of the matrix are complex if $(N-3-2q)^2m^2-4(p-1)<0$. In this case $\frac{1}{2}(N-5-\sqrt{N-2})<q<\frac{N-3}{2}$ which is equivalent to (1.7). Thus if (1.7) holds, then $(1,0)$ becomes a spiral. Using the same argument as in Lemma 3.6, we can show that there is a sequence $\{\alpha_j\}_{j=1}^{\infty}$ $(0<\alpha_1<\alpha_2<\cdots\to\infty)$ such that $\tilde{x}'(\frac{1}{mq}\log\alpha_j)=x'(0)=0$. Since $v'(\frac{\pi}{2})=Am^{-1}x'(0)$, $v'(\frac{\pi}{2})=0$ if $\alpha=\alpha_j$. Since $N-3-2q>0$ and $-\infty<s<\frac{1}{mq}\log\alpha$, (4.2) has the Lyapunov function $I(\tilde{x},\tilde{y})=\frac{\tilde{y}^2}{2}-\frac{\tilde{x}^2}{2}+\frac{\tilde{x}^{p+1}}{p+1}$. We see that $(\tilde{x}(s),\tilde{y}(s))\in\{I<0\}$. Thus, $\tilde{x}>0$ and $v(\theta)$ is positive. We have found infinitely many positive solutions of (4.1) with $v'(\frac{\pi}{2})=0$. $\square$

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